

MARTENS–MUMFORD’S THEOREMS FOR BRILL-NOETHER SCHEMES ARISING FROM VERY AMPLE LINE BUNDLES

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ABSTRACT. Tangent spaces of $V_d^r(L)$ ’s, specific subschemes of C_d arising from various line bundles L on C , are described. Then we proceed to prove Martens theorem for these schemes, by which we determine curves C , which for some very ample line bundle L on C and some integers r and d with $d \leq h^0(L) - 2$, the scheme $V_d^r(L)$ might attain its maximum dimension.

Keywords: Martens–Mumford’s Theorems; Symmetric Products; Very Ample Line Bundle.

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1. Introduction

For a smooth projective algebraic curve C of genus g let C_d denote d -th symmetric product of C . For non-negative integers r, d with $0 \leq 2r \leq d$, the closed subscheme $C_d^r \subset C_d$ parameterizes the locus of divisors of degree d and with space of global sections of dimension at least $r + 1$, on C . The scheme C_d^r can be described locally as the locus of divisors for which the rank of the Brill-Noether matrix does not exceed $d - r$. This globalizes to give the well known scheme structure on C_d^r , as the $(d - r)$ -th determinantal scheme of the morphism:

$$u_* : \Theta_{C_d} \rightarrow \Theta_{\text{Pic}^d(C)}.$$

See [2, Chapter 4] for details and notations.

For an arbitrary line bundle L on C this construction can be generalized to give similar subschemes of C_d , denoted by $V_d^r(L)$, and the usual Brill-Noether schemes, C_d^r ’s, are special cases when L is substituted by the canonical line bundle. See [2, Chapter VII]. The schemes, $V_d^r(L)$ ’s, have been re-appeared recently in [1], where the authors study Koszul Cohomologies of Curves. As a byproduct of their interesting paper, they prove:

Lemma 1.1. (*Aprodu-Sernesi*) *If L is a line bundle on C and $d \geq 4$, then $V_d^1(L)$ is non-empty and of pure dimension $d - 2$.*

Lemma 1.1, although is not one of main themes of paper [1], basically is the starting point of our research. We study the projective geometric aspects of $V_d^r(L)$'s. First in section 3, we describe the tangent spaces of $V_d^r(L)$. As well $V_d^r(L)$ is described as intersection of $V_d^r(H)$'s, where H moves on the set of sub-line bundles of L .

One of the essential ingredients in Brill-Noether Theory is the Martens, as well as Mumford's, Theorem. See Theorems 2.1 and 2.2. We prove Martens theorem for $V_d^r(L)$'s in section 4, Theorem 4.2. This generalizes Theorem 2.1 when K_C is substituted by an arbitrary very ample line bundle L . An existing proof of Martens theorem for $L = K_C$, as it can be found in [2], uses tangent spaces of C_d^r 's together with the fact that C admits a finite number of theta charactersitics. The last part of this proof is not applicable when L is different from the canonical line bundle. To overcome to this obstacle, when C is non-hyper elliptic, we take a different approach which is based on taking an incidence correspondence and counting dimensions, see Theorem 4.2. As a byproduct we reobtain the dimension part of [1, Lemma 2.1] when L is very ample.

During theorem 4.6 we extend the well known Mumford's Theorem to $V_d^r(L)$'s. While we adopt a part of proof of the Mumford's Theorem somehow, our proof is essentially different from existing proof of Mumford's Theorem. Actually Lemmas 4.4 and 4.5, as our basic tool, furnish the way to prove theorem 4.6.

Remember that Keem's Theorem determines curves C , which for them under some circumstances $\dim C_d^r \geq d - r - 2$, to be 4-gonal. See for example [2] or [5]. Through remark 4.7, we see that occurance of two extra type of curves in Theorem 4.6 goes back originally to Keem's Theorem.

During, we follow notations of [2]. Particularly we denote by $V_d^1(L)$ what they denote by $V_{r-q+2}^{r-q+1}(L)$ or by $V_r^{r-1}(L)$ in [1].

2. Preliminaries and backgrounds

For a smooth projective algebraic curve C , let π_1 and π_2 be the projections from $C \times C_d$ to C and C_d respectively. Then for a line bundle L on C , the coherent sheaf

$$\mathcal{L} := (\pi_2)_*(\mathcal{O}_\Delta \otimes \pi_1^*L),$$

where $\Delta \subset C \times C_d$ is the universal divisor of degree d , is a vector bundle of rank d on C_d . Moreover for $D \in C_d$ we have the identifications

$$\mathcal{L}_D \cong H^0(C, L \otimes \mathcal{O}_D) \cong H^0(C, L/L(-D)).$$

The natural map $\pi_1^*L \rightarrow \mathcal{O}_\Delta \otimes \pi_1^*L$ induced by restriction on Δ , which is a map of vector bundles on $C \times C_d$, pushes forward via π_2 to a map

of vector bundles on C_d :

$$f_L : H^0(C, L) \otimes \mathcal{O}_{C_d} \rightarrow \mathcal{L}.$$

Assuming L to be a line bundle of degree δ and the space of global sections of dimension $s + 1$, the map f_L would be a map of vector bundles of ranks $s + 1$ and d respectively. For a non negative integer r set

$$V_d^r(L) := \{D \in C_d \mid \text{rk}(f_L)_D \leq d - r\}.$$

The subscheme $V_d^r(L) \subset C_d$ parameterizes those effective divisors of degree d on C that impose at most $d - r$ conditions on $|L|$, as well it is expected to be of dimension $d - r(s + 1 - d + r)$.

For $L = K_C$, as it is commonly used in literature, the scheme $V_d^r(K_C)$ will be denoted by C_d^r . Let

$$\begin{aligned} \alpha : C_d &\rightarrow \text{Pic}^d(C) \\ D &\mapsto \mathcal{O}(D), \end{aligned}$$

be the Abel map and set $W_d^r(C) := \alpha(C_d^r)$. Using the notion of Poincare line bundle, the subset $W_d^r(C) \subset \text{Pic}^d(C)$ admits the structure of a closed subscheme. Furthermore Martens Theorem gives an upper bound for $\dim W_d^r(C)$, as well as Mumford’s Theorem classifies curves C for which $\dim W_d^r(C)$ attains its maximum value for some integers r and d .

Theorem 2.1. *(Martens) Let C be a smooth curve of genus $g \geq 3$. Let d be an integer such that $2 \leq d \leq g - 1$ and let r be an integer such that $0 < 2r \leq d$. Then if C is non hyper-elliptic, every component of $W_d^r(C)$ has dimension at most equal to $d - 2r - 1$. If C is hyper-elliptic, then $\dim W_d^r(C) = d - 2r$.*

Theorem 2.2. *(Mumford) Let C be a smooth non-hyper elliptic curve of genus $g \geq 4$. Suppose that there exist integers r and d such that $2 \leq d \leq g - 2$, $d \geq 2r > 0$ and a component X of $W_d^r(C)$ with $\dim X = d - 2r - 1$. Then C is either trigonal, bielliptic or a smooth plane quintic.*

See [2] and [8] for proof of Theorems 2.1 and 2.2. Based on Mumford’s Theorem, we call a curve C of Mumford’s type if either it is bielliptic, trigonal or a smooth plane quintic.

3. Tangent Space computations for $V_d^r(L)$

Let $\{\omega_1^L, \omega_2^L, \dots, \omega_{s+1}^L\}$ be a basis for $H^0(C, L)$ and $\phi^L : C_d \rightarrow M_{d \times (s+1)}$ be the map defined by

$$\phi^L\left(\sum_{i=1}^{i=d} q_i\right) = (\omega_t^L(q_i))_{i,t}$$

where $M_{d \times (s+1)}$ is the space of d by $(s+1)$ - matrices. For $D \in C_d$ setting $A = \phi^L(D)$ the restriction map $\alpha_L : H^0(C, L) \rightarrow H^0(C, L \otimes \mathcal{O}_D)$, is represented by A . As a consequence of this fact; for $\nu \in T_D(C_d)$ one might identify $\phi_*^L(\nu) \cdot \ker(A)$ with $\beta_L(\nu \otimes H^0(C, L(-D)))$, where β is the cup product homomorphism

$$H^0(C, \mathcal{O}_D(D)) \otimes H^0(C, L(-D)) \rightarrow H^0(C, L \otimes \mathcal{O}(D)).$$

Lemma 3.1. (a) If D belongs to $V_d^r(L) \setminus V_d^{r+1}(L)$, the tangent space to $V_d^r(L)$ at D is

$$T_D(V_d^r(L)) = (\text{Im}(\alpha_L \mu_0^L))^\perp$$

where μ_0^L is the cup product map

$$\mu_0^L : H^0(C, \mathcal{O}(D)) \otimes H^0(C, L(-D)) \rightarrow H^0(C, L).$$

(b) If $D \in V_d^{r+1}(L)$ then $T_D(V_d^r(L)) = H^0(C, L \otimes \mathcal{O}_D)$. Particularly, if $V_d^r(L)$ has the expected dimension and $d < s + 1 + r$, then $D \in \text{Sing}(V_d^r(L))$.

Proof. (a) This is a repetition of discussions in pages 161 – 162 of [2].
 (b) For $D \in C_d$, we have $D \in V_d^{r+1}(L)$ if and only if $\phi^L(D) \in M_{d \times (s+1)}(r+1)$. Now the equality $T_{\phi^L(D)} M_{d \times (s+1)} = M_{d \times (s+1)}$, which leads to the assertion, is a well known fact. See [2, Chapter II-Section 2]. \square

Theorem 3.2. The scheme $V_d^r(L)$ is smooth at $D \in V_d^r(L) \setminus V_d^{r+1}(L)$ and has the expected dimension $d - r \cdot (s + 1 - (d - r))$ if and only if μ_0^L is injective.

Proof. Since $\text{Ker}(\alpha_L) = H^0(C, L(-D)) \subset \text{Im}(\mu_0^L)$ one has

$$\begin{aligned} \dim T_D[V_d^r(L)] &= d - (\dim \text{Im}(\mu_0^L) - \dim \text{Ker}(\alpha_L)) \\ &= d - r \cdot h^0(C, L(-D)) + \dim \text{Ker} \mu_0^L \\ &= d - r \cdot (s + 1 - (d - r)) - r \cdot (d - r - \dim \text{Im}(\alpha_L)) + \dim \text{Ker} \mu_0^L. \end{aligned}$$

This implies the assertion. \square

Lemma 3.3. *For a line bundle L and integers r, d with $h^0(L) > d - r + 1$, we have:*

$$V_d^r(L) = \bigcap_{H \subseteq L} V_d^r(H),$$

where H moves on the set of sub-line bundles of L .

Proof. If D is a point of $V_d^r(H)$, then

$$\dim(\operatorname{Im}(e_H^D)) = \operatorname{rank}(e_H^D) \leq d - r.$$

If H is a sub-line bundle of the line bundle L , then a diagram chasing shows $\operatorname{Im}(e_H^D) \subseteq \operatorname{Im}(e_L^D)$. This implies that $V_d^r(L) \subseteq V_d^r(H)$.

If a divisor $D \in C_d$ belongs to

$$\left[\bigcap_{H \subseteq L} V_d^r(H) \right] \setminus V_d^r(L),$$

then for each $p \in C$ we have $h^0(L(-p)(-D)) = h^0(L(-D))$, which means that; any point of C is a base point for $L(-D)$. Therefore any global section of $L(-D)$ vanishes everywhere on C , so $H^0(L(-D)) = 0$. This together with the fact that for $p \in C$ we have $D \in V_d^r(L(-p))$, implies $h^0(L) \leq d - r + 1$, which is absurd by our hypothesis $h^0(L) > d - r + 1$. □

4. MARTENS–MUMFORD’S THEOREMS FOR $V_d^r(L)$

In this section we assume that L is a very ample line bundle on C . Therefore $\psi_L : C \rightarrow \mathbf{P}(H^0(C, L)) := \mathbf{P}_L$, the map induced by L , is an embedding. Repeating the proof of [1, Lemma 2.2] we obtain:

Lemma 4.1. *For a very ample line bundle L on C and an integer d with $d \geq 4$, if $V_d^r(L) \neq \emptyset$, then no irreducible component of $V_d^r(L)$ is contained in $V_d^{r+1}(L)$.*

A direct consequence of Lemma 4.1 is that the locally closed subscheme

$$S_d^r(L) := V_d^r(L) \setminus V_d^{r+1}(L)$$

is dense in any irreducible component of $V_d^r(L)$.

Theorem 4.2. (a) *Let C be a hyper-elliptic curve and L a line bundle on C with the space of global sections of dimension $s + 1$. Assume moreover that $d \leq s + 1$. Then $V_d^r(L)$ is empty or irreducible of dimension $d - r$ according to whether $d < 2r$ or $2r \leq d$, respectively.*

(b) *If C is non hyper-elliptic and L a very ample line bundle on C with $d \leq h^0(L) - 1$, then every component of $V_d^r(L)$, has dimension at most equal to $d - r - 1$.*

Proof. (a) Without loss of generality we reduce to the case that L is base point free, so $L = sg_d^1$. If $D \in C_d$ then $D \in V_d^r(L) \setminus V_d^{r+1}(L)$ if and only if $h^0(L(-D)) = s + r + 1 - d$. If $h^0(D) = \bar{r} + 1$ then $D = \bar{r}g_2^1 + q_1 + \cdots + q_{d-2\bar{r}}$ for some $q_1, \dots, q_{d-2\bar{r}}$ on C and $\bar{L} = (s - \bar{r})g_2^1$. Consider that the divisor $\phi_{\bar{L}}(q_1) + \cdots + \phi_{\bar{L}}(q_{d-2\bar{r}})$ on the rational normal curve $\phi_{\bar{L}}(C) \subset \mathbf{P}(H^0((s - \bar{r})g_2^1))$, spans a $\mathbf{P}^{d-2\bar{r}-1}$ in $\mathbf{P}(H^0((s - \bar{r})g_2^1))$. Therefore we have $h^0(L(-D)) = s + \bar{r} + 1 - d$. An immediate consequence of this observation is that $V_d^r(L) = \emptyset$ provided $2r > d$ and $V_d^r(L) \setminus V_d^{r+1}(L) = C_d^r \setminus C_d^{r+1}$ for $2r \leq d$. This concludes (a).

To prove (b), consider an incidence correspondence \mathcal{H} as

$$\mathcal{H} = \{(H, D) : D \subset H \cap C_L\} \subset (\mathbf{P}_L)^* \times C_d,$$

where H is a hyperplane in \mathbf{P}_L . Assuming $V_d^r(L) \neq \emptyset$, let W be an irreducible component of $V_d^r(L)$ of maximal dimension. Using Lemma 4.1, we have $W = \overline{(V_d^r(L) \setminus V_d^{r+1}(L))} \cap W$. Consider the subscheme Σ of \mathcal{H} as:

$$\Sigma := \pi_2^{-1}((V_d^r(L) \setminus V_d^{r+1}(L)) \cap W) \subset \mathcal{H},$$

where π_2 is the second projection from $(\mathbf{P}_L)^* \times C_d$ composed with the inclusion.

For $D \in C_d$ it is easy to see that either $|D| \cap V_d^r(L) = \emptyset$ or $|D| \subseteq V_d^r(L)$. This together with the geometric Riemann-Roth Theorem, imply that a point of $V_d^r(L) \setminus V_d^{r+1}(L)$ lies in a $\mathbf{P}^s \subset \mathbf{P}_L$, where $s = h^0(L) - 1 - h^0(L(-D)) = d - r - 1$. Therefore the generic fiber of $\Sigma \rightarrow (V_d^r(L) \setminus V_d^{r+1}(L)) \cap W$ is a \mathbf{P}^m , where $m = h^0(L) - d + r - 1$. So we obtain

$$\dim(\Sigma) = \dim((V_d^r(L) \setminus V_d^{r+1}(L)) \cap W) + h^0(L) - d + r - 1.$$

Consider now that the projection on the second factor is a finite to one and non-surjecting map by the general position Lemma. This consideration implies that $\dim(\Sigma) \leq h^0(L) - 2$. Summarizing we get

$$\dim(V_d^r(L)) = \dim(V_d^r(L) \setminus V_d^{r+1}(L)) \leq d - r - 1.$$

□

Using Theorem 4.2, we recover the dimension part of [1, Lemma 2.1] for very ample line bundles.

Corollary 4.3. *Assume that L is a very ample line bundle on C with $h^0(L) = d + 1 \geq 4$. Then $V_d^1(L)$, if non empty, is of dimension $d - 2$.*

To obtain Mumford's Theorem on $V_d^r(L)$'s, Theorem 4.6, we need the next couple of lemmas.

Lemma 4.4. *Assume that L is a very ample line bundle on C such that for some integers $r \geq 2$ and d with $d \leq h^0(L) - 2$ we have $\dim V_d^r(L) = d - r - 1$, then $V_{d-1}^{r-1}(L)$ is of dimension $d - r - 1$ too.*

Proof. We assume that $V_d^r(L)$ is irreducible, since otherwise we can substitute it with an irreducible component. As we mentioned in proof of Theorem 4.2, for $D \in C_d$ we have either $|D| \cap V_d^r(L) = \emptyset$ or $|D| \subseteq V_d^r(L)$. For general $q \in C$ from the equality

$$V_d^r(L) = \bigcup_{p \in C} (p + C_{d-1}) \cap V_d^r(L),$$

we obtain $\dim(q + C_{d-1}) \cap V_d^r(L) = \dim V_d^r(L) - 1$. In fact for general $q \in C$ the equality $V_d^r(L) = (q + C_{d-1}) \cap V_d^r(L)$ implies that q , being a general point of C , is a base point of $|D|$ for each $D \in V_d^r(L)$. In other words each global section σ of $\mathcal{O}(D)$ vanishes at q . Therefore σ vanishes everywhere on C . So $H^0(D) = 0$ which by effectivity of D is absurd, proving the equality $\dim(q + C_{d-1}) \cap V_d^r(L) = \dim V_d^r(L) - 1$.

To establish the Lemma, consider that removing q from the series in $V_d^r(L)$ we obtain $(1 + (d - r - 1) - 1)$ -dimensional family of divisors \bar{D} belonging to $V_{d-1}^{r-1}(L)$. This together with Theorem 4.2, imply the assertion. \square

Lemma 4.5. *Let L be a base point free line bundle on C such that for general $p \in C$, the line bundle $L(-p)$ is very ample. Assume moreover that V is an irreducible component of $V_d^1(L)$ and $d = h^0(L) - 2 \geq 4$. Then for general $p \in C$, no irreducible component W_p of $V_d^1(L(-p))$ coincides on V .*

Proof. We prove that a general divisor D in V fails to be a general member of a component of $V_d^1(L(-p))$.

For each $D \in V$, there exists $p \in C$ such that the subset $(p + C_{d-1}) \cap V_d^1(L)$ is non-empty. For general D in V , since $\dim V \geq 2$, this p has to move in an open subset of C . Otherwise for general $D \in V$ removing p from the divisors of V we obtain $\dim V_{d-1}^1(L(-p)) = d - 2$, contradicting Lemma (2.1) of [1].

If a general divisor D in V turns to be a general member of a component W_p of $V_d^1(L(-p))$; then for general $q \in C$ the divisors of type $D - p + q$, belonging to $V_d^1(L(-p))$, lie on W_p . This by genericity of D and q is impossible, which completes the proof of Lemma. \square

Theorem 4.6. *Assume that C is a smooth projective non-hyper elliptic curve of genus g with $g \geq 9$. If for some very ample line bundle L on*

C there exist integers r, d with $d \leq h^0(L) - 2$ and a component X of $V_d^r(L)$ with $\dim X = d - r - 1$, then C is one of the following type:

- A bi-elliptic curve,
- A 3-gonal curve,
- A 4-gonal curve, or
- A space curve of degree 7.

Proof. Everywhere in this proof to simplify the notations, we shall write $V_d^r(L)$ instead of X . Consider first that using Lemma 4.4 we can reduce to the case $r = 1$ and we have $\dim V_d^1(L) = d - 2$. Let d be the minimum integer for which for some line bundle K one has $\dim V_d^1(K) = d - 2$. Assume moreover that L is a very ample line bundle with minimum $h^0(L)$ among those very ample line bundles H , which for them $V_d^1(H)$ is of dimension $d - 2$. Under these circumstances, for a general $p \in C$ either $L(-p)$ fails to be very ample or $d \geq h^0(L(-p)) - 1$. The latter case implies $h^0(L) = d + 2$.

In the first case; for a general $p \in C$ there are $q_1, q_2 \in C$ such that $h^0(L(-p)(-q_1 - q_2)) \geq h^0(L(-p)) - 1 = h^0(L) - 2$. Therefore $h^0(L) - h^0(L(-p - q_1 - q_2)) \leq 2 = 3 - 1$, so $p + q_1 + q_2 \in V_3^1(L)$. This, according to Theorem 4.2, asserts $\dim V_3^1(L) = 1$. Therefore $d \leq 3$.

Using very ampleness of L we have $V_2^1(L) = \emptyset$, excluding the case $d = 2$.

We proceed analyzing the case $d = 3$. If for a $D \in V_3^1$ one has $h^0(D) = 2$ then C is trigonal, while if for each $D \in V_3^1$ we had $h^0(D) = 1$ then for D_1 and D_2 in V_3^1 , we'll have $h^0(D_1 + D_2) \in \{2, 3\}$. Therefore two subcases occur.

If $h^0(D_1 + D_2) = 2$ then $h^0(K(-D_1 - D_2)) = g - 5$. For general points p_1, p_2, \dots, p_{g-8} on C we have $h^0(K(-2D_1 - 2D_2)(-\Sigma p_i)) \geq 1$, therefore $|D_1 + D_2|$ is contained in $|K(-D_1 - D_2)(-\Sigma p_i)| = |M|$. Let $\phi : C \rightarrow \mathbf{P}^3$ be the morphism defined by M and consider that the morphism $\phi_{|D_1 + D_2|} : C \rightarrow \mathbf{P}^1$, defined by $|D_1 + D_2|$, is obtained by composing ϕ with a projection from a line L in \mathbf{P}^3 which we can assume it pass through a smooth point of $\phi(C)$. Therefore

$$(\deg \phi)(\deg \phi(C) - 1) = 6.$$

Up to the last equality either $\deg \phi = 1$ in which case $\phi(C)$, as well as C , is a space septic or; $\deg \phi = 2$ in which case $\phi(C)$ is a space quartic which has to be a normal elliptic curve. Therefore C is a double covering of a normal elliptic space curve, which is the same as to be bi-elliptic. Lastly $\deg \phi = 3$ and $\phi(C)$ is a space cubic curve which has to be a rational normal curve. Therefore C is a triple covering of a rational normal space curve, which means that C is trigonal.

If we are in the second subcase; i.e. for general $D_1, D_2 \in V_3^1$ we have $h^0(D_1 + D_2) = 3$, then $\dim C_6^2 \geq 2$, so

$$\dim W_6^2 \geq 0 = 6 - 2 \times 2 - 2.$$

Applying Keem’s Theorem, see [2, page 200], and its extension by Coppens to the cases $g = 9, 10$ in [4], imply that C is a 4-gonal curve.

Assume finally that $d = h^0(L) - 2$. If for general $p \in C$ the line bundle $L(-p)$ fails to be very ample or if $d = 3$, then our position is the same as previous case.

We exclude the other case. Assume that for general $p \in C$ the line bundle $L(-p)$ is very ample with $d \geq 4$ and consider that if V is an irreducible component of $V_d^1(L)$ with maximal dimension $(d - 2)$, then for each $p \in C$ using Lemma (2.1) from [1], an irreducible component of $V_d^1(L(-p))$ coincides on V . This, according to Lemma 4.5, is absurd. \square

Remark 4.7. (a) Very ampleness in Theorem 4.2 is necessary. To see this; for $p \in C$ set $L = K(p)$ and consider that for general points q_1, q_2 on C and $D \in C_{g-1}^1$, effective divisors linearly equivalent to divisors of type $D + q_1 + q_2 - p$ belong to $V_g^1(L)$. So $\dim V_g^1(L) \geq g - 1 > g - 1 - 1$.

(b) For curves of Mumford’s type, the bound in Theorem 4.6 is achieved when $L = K_C$, as it is well known in the literature. For very ample canonical sub-line bundles $L \subseteq K_C$, since by Lemma 3.3 $V_d^r(L)$ contains C_d^r , then $\dim V_d^r(L)$ attains the bound of Theorem 4.6.

(c) If C is bielliptic and L is a very ample sub-line bundle of K_C , then Lemma 3.1 together with Theorem 4.2, imply maximum dimensionality of $V_d^2(L)$, i.e. $\dim(V_d^2(L)) = d - 3$.

If L is a very ample line bundle which is not a sub-line bundle of the canonical bundle, denote by $\epsilon : C \rightarrow E$ the elliptic involution and assume that L enjoys from the property that for some $p, q \in E$ one has

$$h^0(L) - h^0(L(-p_1 - p_2 - q_1 - q_2)) \leq 3$$

where $\epsilon^{-1}(p) = \{p_1, p_2\}$ and $\epsilon^{-1}(q) = \{q_1, q_2\}$. Then for $R_1, \dots, R_t \in C$ we have $p_1 + p_2 + q_1 + q_2 + R_1 + \dots + R_t \in V_{t+4}^1(L)$ and $\dim(V_{t+4}^1(L)) = t + 2$. Additionally for some general $\gamma \in E$, with an extra prescribed assumption

$$h^0(L) - h^0(L(-p_1 - p_2 - q_1 - q_2 - \gamma_1 - \gamma_2)) \leq 4$$

for which $\epsilon^{-1}(\gamma) = \{\gamma_1, \gamma_2\}$, the divisors of type

$$p_1 + p_2 + q_1 + q_2 + \gamma_1 + \gamma_2 + R_1 \cdots + R_t$$

belong to $V_{t+6}^2(L)$. This again imply maximum dimensionality of $V_{t+6}^2(L)$, i.e. $\dim(V_{t+6}^2(L)) = t + 3$.

(d) Assume that C is a 4-gonal curve with $p \in C$ a base point of $K(-g_4^1)$. Consider the very ample line bundle $L = K(-p)$ and observe that; divisors of type $g_4^1 + q_1 + \cdots + q_t$ belong to $V_{t+4}^2(L)$. Therefore $t + 1 \leq \dim V_{t+4}^2(L)$. This using Theorem 4.2 implies $\dim V_{t+4}^2(L) = t + 1$.

(e) If C is a space curve of degree 7, then projecting from an smooth point of C to \mathbf{P}^2 we obtain a plane sextic, which is singular. Therefore C is birational to a 4-gonal curve. This, by example (d) implies that $\dim V_d^1(L)$ might attain its maximum value.

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